FUNDAMENTAL EQUATIONS

A fluid in motion is subjected to several forces which results in the variation of accelerations and the energies involved in the phenomenon of the fluid. As such in the study of fluid motion the forces and energies that are involved in the flow are required to be considered. This aspect of fluid motion is known as dynamics of fluid flow.

The various forces acting on the fluid mass may be classified as body or volume forces, surface forces and line forces.

The body or volume forces are the forces which are proportional to the volume of the body. Examples: weight, centrifugal force etc,.

The surface forces are the forces which are proportional to the surface area. Examples: pressure force, shear force, force of compressibility etc,.

The line forces are the forces which are proportional to the length. Examples: surface tension.

Forces acting on fluid in motion

The various forces that may influence the motion of a fluid are due to gravity, pressure, viscosity, turbulence, surface tension and compressibility. The gravity force F_g is due to the weight of the fluid and is equal to 'Mg'. The pressure force F_p is exerted on the fluid mass if there exists a pressure gradient between two points in the direction of the flow. The viscous force F_v is due to the viscosity of the flowing fluid and thus exists in the case of all real fluids. The turbulent force F_t is due to the turbulence of the flow. In the turbulent flow the fluid particles move from one layer to other and therefore, there is a continuous momentum transfer between adjacent layers, which results in developing additional stresses (called Reynolds stresses) for the flowing fluid. The surface tension force F_s is due to the cohesive property of the fluid mass. It is considered when the depth of the flow is extremely small. The compressibility force F_e is due to the elastic property of the fluid and it is important only either for compressible fluids or in the case of flowing fluids in which the elastic properties of the fluid s are significant.

If a certain mass of fluid in the motion is influenced by all the above mentioned forces, then according to Newton's second law of motion the following equations may be written,

$$Ma = F_g + F_p + F_v + F_t + F_s + F_e \tag{1}$$

In most of the problems of fluid in motion the surface tension forces and compressibility forces are not significant. Hence these forces may be neglected. Then equation (1) becomes,

$$Ma = F_g + F_p + F_v + F_t \tag{1.2}$$

Equation (1.2) is known as **Reynold's equation of motion** which is useful in the analysis of the turbulent flows.

Further for laminar or viscous flows the turbulent forces also become less significant and hence these may be neglected. The equation (1.2) may then be modified as

$$Ma = F_g + F_p + F_v \tag{1.3}$$

Equation (1.3) is known as **Navier-Stokes's equation of motion** which is useful in the analysis of viscous flow.

Further if the viscous forces are also of little significance in the problems of fluid flows, then these forces may also be neglected. The viscous forces will become insignificant if the flowing fluid is an ideal fluid. Even, in the case of real fluids also the viscous forces may be considered to be insignificant if the viscosity of the flowing fluid is small. In such cases, the above equations may be further modified as

$$Ma = F_g + F_p \tag{1.4}$$

Further by resolving the various forces and accelerations along x, y, and z directions the following equations of motion may be obtained.

$$Ma_x = F_{gx} + F_{px}$$
$$Ma_y = F_{gy} + F_{py}$$
$$Ma_z = F_{gz} + F_{pz}$$

The above equations are known as **Euler's equations of motion**.

Euler's equations of motion

This is the equation of motion in which the forces due to gravity and pressure are taken into consideration. It is derived by considering the motion of a fluid element along a stream line as:

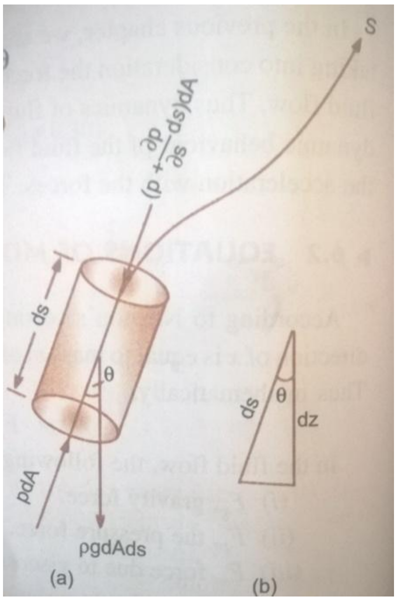
Consider a stream line in which flow is taking place in *S*-direction as shown in Fig. . Consider a cylindrical element of cross section *dA* and length *dS*. The forces acting on the cylindrical element are:

1. Pressure forces *pdA* in the direction of flow

2. Pressure force $(p + \frac{\partial P}{\partial S} dS)\, dA$ opposite to the direction of the flow.

3. Weight of the element ρg*dAds.*

Let Θ is the angle between the direction of the flow and the line of action of the weight of the element.



The resultant force on the fluid element in the direction *S* must be equal to the product of mass of fluid element and acceleration in the direction *S*.

Therefore, $pdA - (p + \frac{\partial P}{\partial S} dS)\, dA - \rho g dAds \cos\Theta = \rho\, dAds \times a_s$ (2)

Where a_s is the acceleration in the direction of *S*.

Now, $a_s = \frac{dv}{dt}$, where *v* is a function of *s* ant *t*.

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v \partial v}{\partial s} + \frac{\partial v}{\partial t} \qquad \left\{ since, \frac{ds}{dt} = v \right\}$$

If the flow is steady, then $\frac{\partial v}{\partial t} = 0$

Therefore, $a_s = \frac{v\partial v}{\partial s}$ and by substituting this value in equation (2) and simplifying the equation we get,

$$-\frac{\partial P}{\partial S} dS\, dA - \rho g dAds \cos\Theta = \rho\, dAds \times \frac{v\partial v}{\partial s}$$

or $$\frac{\partial P}{\rho\, \partial S} + g \cos\Theta + \frac{v\partial v}{\partial s} = 0$$

But from Fig. we have $\cos\Theta = \frac{dz}{ds}$ and substituting this value in the above equation, we get,

$$\frac{\partial p}{\rho} + gdz + vdv = 0 \qquad (3)$$

Equation (3) is known as Euler's equation of motion.

Bernoulli's equation

The equation obtained by integrating Euler's equation of motion is known as Bernoulli's equation.

$$\int \frac{\partial p}{\rho} + \int gdz + \int vdv = constant$$

If flow is incompressible, then ρ is constant and

Therefore, $$\frac{p}{\rho} + gz + \frac{v^2}{2} = constant$$

or $$\frac{p}{\rho g} + z + \frac{v^2}{2g} = constant$$

or $$\frac{p}{\rho g} + \frac{v^2}{2g} + z = constant \qquad (4)$$

Equation (4) represents Bernoulli's equation which represents the energy possessed by the flowing fluid and terms in this equation represents energies per unit weight of the fluid.

$$\frac{p}{\rho g} = pressure\ energy\ per\ unit\ weight\ of\ fluid\ (or)\ pressure\ head$$
$$\frac{v^2}{2g} = Kinetic\ energy\ per\ unit\ weight\ of\ fluid\ (or)\ Kinetic\ head$$
$$z = Potential\ energy\ per\ unit\ weight\ of\ fluid\ (or)\ Potential\ head$$

The energy per unit weight of the fluid is expressed as N.m/N or kg(f) – m /kg(f), that is it has a dimension of length and therefore it is known as head. The sum of the pressure head and potential head is also called as Piezometric head ($\frac{p}{\rho g} + z$).

Assumptions

The following are the assumptions made in the derivation of Bernoulli's equation:

i. Fluid is ideal

ii. Flow is steady

iii. Flow is incompressible

iv. Flow is irrotational.

Thus Bernoulli's equation states that, in a steady, irrotational flow of an incompressible fluid the total energy at any point is constant. If the Bernoulli's equation is applied between any two points in a steady irrotational flow of an incompressible fluid then, we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

For the flow of real fluids, since there is always some energy of the flowing fluid converted into heat due to the viscous and turbulent shear and consequently there is a certain amount of energy loss. Hence for the flow of real fluids, the Bernoulli's equation can be modified as

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L$$

Where h_L represents the loss of energy (or head) between the points under consideration.

Practical application of Bernoulli's equation

Bernoulli's equation along with the continuity equation is commonly used in the solution of the problems of fluid flow. Some of the simple applications of Bernoulli's equation are venturi meter, orifice meter, nozzle meter, pitot tube.

1. The water is flowing through a pipe having diameters 20 cm and 10 cm at sections 1 and 2 respectively. The rate of flow through the pipe is 35 lit/s. The section 1 is 6 m above datum and section 2 is 4 m above datum. If the pressure at section 1 is 39.24 N/cm^2, find the intensity of pressure at section 2.

Ans. Given data:

At section 1, D_1 = 20 cm = 0.2 m

$A_1 = \frac{\pi}{4} 0.2^2 = 0.0314$ m^2 ; z_1 = 6 m

P_1 = 39.24 N/cm^2 = 39.24 x 10^4 N/m^2

At section 2, D_2 = 10 cm = 0.1 m

$A_2 = \frac{\pi}{4} 0.1^2 = 0.00785$ m^2 ; z_2 = 4 m

P_2 = ?

Rate of flow Q= 35 lit/s = 0.035 m^3/s

From continuity equation, $Q = A_1V_1 = A_2V_2$

Therefore $V_1 = \frac{Q}{A_1} = \frac{0.035}{0.0314} = 1.11\ m/s$

$$V_2 = \frac{Q}{A_2} = \frac{0.035}{0.00785} = 4.456\ m/s$$

Applying Bernoulli's equation at sections 1 and 2,

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

$$\frac{39.24\ x\ 10000}{1000\ x\ 9.81} + \frac{1.11^2}{2\ x\ 9.81} + 6 = \frac{p_2}{1000\ x\ 9.81} + \frac{4.456^2}{2\ x\ 9.81} + 4$$

or P_2 = 41.051 x 9810 N/m^2

= 41.051 x 9810 x 10^{-4} N/cm^2

= **40.27 N/cm^2**

2. A pipe of diameter 400 mm carries water at a velocity of 25 m/s. The pressures at the points A and B are given as 29.43 N/cm^2 and 22.563 N/cm^{2}respectively while the datum head at A and B are 28m and 30m. Find the loss of head between A and B.

Ans. Given data: D = 400 mm = 0.4 m **;** $V = V_A = V_B = 25\ m/s$

At point A,

Z_A = 28 m

P_A = 29.43 N/cm^2 = 29.43 x 10^4 N/m^2

Total energy at A,

$$E_A = \frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A$$

$$= \frac{29.43\ x\ 10000}{1000\ x\ 9.81} + \frac{25^2}{2\ x\ 9.81} + 28$$

= 30 + 31.85 + 28 = 89.85 m

At point B, Z_B = 30 m

P_B = 22.563 N/cm^2 = 29.43 x 10^4 N/m^2

Total energy at B,

$$E_B = \frac{p_B}{\rho g} + \frac{v_B^2}{2g} + z_B$$

$$= \frac{22.563\ x\ 10000}{1000\ x\ 9.81} + \frac{25^2}{2\ x\ 9.81} + 30$$

= 23 + 31.85+ 30 = 84.85 m

Therefore loss of energy = Loss of head = E_A - E_B = 89.85 -84.85 = **5.0 m**

Exercise problems

1. Water is flowing through a pipe of 5 cm diameter under a pressure of 29.43 N/cm^2 with mean velocity of 2.0 m/s. Find the total head or total energy per unit weight of water at a cross section, which is 5 m above the datum line.

2. A pipe, through which water is flowing, is having diameters, 20 cm and 10 cm at the cross sections 1 and 2 respectively. The velocity of water at section 1 is given 4 m/s. Find the velocity head at sections 1 and 2 and also rate of discharge.

3. Water is flowing through a pipe having 300 mm and 200 mm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 24.525 N/cm^2 and the pressure at the upper end is 9.81 N/cm^2. Determine the difference in datum head if the rate of flow through the pipe is 40 lit/s.

4. The water is flowing through a tapering pipe of length 100 m having diameters 600 mm at the upper end and 300 mm at the lower end, at the rate of 50 lit/sec. The pipe has a slope of 1 in 30. Find the pressure at the lower end if pressure at the higher level is 19.62 N/cm^2.

5. A pipe line carrying oil of specific gravity 0.87, changes in diameter from 200 mm diameter at a position A to 500 mm diameter at a position B which is 4 meters at a higher level. If the pressures at A and B are 9.81 N/cm^2 and 5.886 N/cm^2 respectively and the discharge is 20 lit/sec. Determine the loss of head and direction of flow.

Venturimeter

It is a device used for measuring the rate of flow of a fluid flowing through a pipe, based on the principle of Bernoulli's equation. It consists of three parts:

i. Short converging part
ii. Throat
ii. Divergent part

Consider a venturimeter fitted in a horizontal pipe through which a fluid is flowing, as shown in Fig.

Let d_1 = diameter at inlet or at section 1

P_1 = pressure at section 1

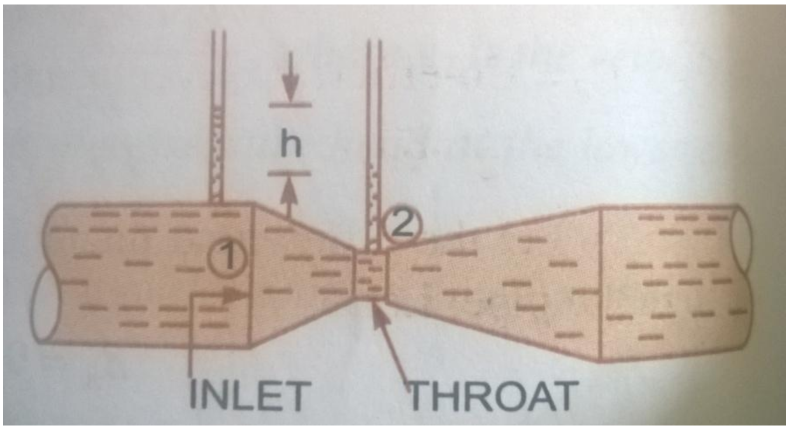
V_1 = velocity of fluid at section 1

a_1 = area at section 1 and

d_2, P_2, V_2, and a_2 are corresponding values at section 2.

On applying Bernoulli's equation at sections 1 and 2, we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$



As pipe is horizontal, $z_1 = z_2$ and $\frac{P_1 - P_2}{\rho g} = h$, where h is the manometric head which is equals to difference of pressure heads between two sections.

Therefore the Bernoulli's equation can be modified as $h = \frac{V_1^2}{2g} - \frac{V_2^2}{2g}$

Now applying continuity equation at sections 1 and 2

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad V_1 = \frac{V_2 a_2}{a_1}$$

Substituting the value of V_1 in the above equation and simplifying, we get

$$V_1 = \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$$

Therefore discharge, $Q = a_2 v_2$

$$Q = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$$

This equation gives the discharge under ideal conditions and is called as theoretical discharge. Actual discharge will be less thsn the theoretical discharge and can be calculated as

$$Q_{actual} = C_d \; x \; Q_{theoretical}$$

Where C_d is known as coefficient of discharge and its value is less than unity.

Value of manometric head given by differential manometer

Case I.

Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Then, $h = x\left[\frac{S_m}{S_f} - 1\right]$.

Case II.

Let the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe. Then, $h = x\left[1 - \frac{S_m}{S_f}\right]$.

Where S_m = Specific gravity of the manometric liquid

S_f = Specific gravity of the liquid flowing through pipe

x = difference of the manometric liquid and flowing liquid.

1. An oil of specific gravity 0.8 is flowing through a ventuirmeter having inlet diameter 20 cm and throat diameter 10 cm. The oil mercury differential manometer shows a reading of 25cm. Calculate the discharge of oil through the horizontal ventuirmeter. Take C_d = 0.98.

Ans. Given data: Sp. gr. of oil, S_f = 0.8

Sp. gr. of mercury, S_m = 13.6

Reading of differential manometer = x = 25 cm

Therefore, difference of pressure head = $h = x\left[\frac{S_m}{S_f} - 1\right] = 25\left[\frac{13.6}{0.8} - 1\right]$ = 400 cm of oil.

Diameter at inlet = d_1 = 20 cm and $a_1 = \frac{\Pi}{4} d_1^2 = \frac{\Pi}{4} 20^2$ = 314.16 cm^2

Diameter at throat = d_2 = 10 cm and $a_2 = \frac{\Pi}{4} d_2^2 = \frac{\Pi}{4} 10^2$ = 78.54 cm^2

The discharge is given by the equation $Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$

$$= 0.98 \frac{314.16 \, x \, 78.54}{\sqrt{314.16^2 - 78.54^2}} \sqrt{2x981x400}$$

= 70465 cm^3/s = **70.465 lit/s**

2. A horizontal ventuirmeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of oil of specific gravity 0.8. The discharge of oil through ventuirmeter is 60 litres/s.. Find the reading of the oil mercury differential manometer. Take C_d = 0.98.

Ans. Given data: Diameter at inlet = d_1 = 20 cm and $a_1 = \frac{\Pi}{4} d_1^2 = \frac{\Pi}{4} 20^2$ = 314.16 cm^2

Diameter at throat = d_2 = 10 cm and $a_2 = \frac{\Pi}{4} d_2^2 = \frac{\Pi}{4} 10^2$ = 78.54 cm^2

The discharge Q = 60 lit/s = 60 x 1000 cm^3/s

But the discharge is given by the equation $Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$

$$60x1000 = 0.98\ \frac{314.16\ x\ 78.54}{\sqrt{314.16^2 - 78.54^2}}\sqrt{2x981xh}$$

On solving the above equation, we get h = 289.98 cm of oil.

But $$h = x\left[\frac{S_m}{S_f} - 1\right]$$

$$289.98 = x\left[\frac{13.6}{0.8} - 1\right] = 16x$$

Then x = 18.12 cm

Therefore, reading of oil - mercury differential manometer = 18.12 cm

Exercise problems

1. A horizontal ventuirmeter with inlet and throat diameters 30 cm and 15 cm respectively is used to measure the flow of water. The reading of the differential manometer connected to the inlet and throat is 20 cm of mercury. Determine the rate of flow. Take C_d = 0.98.

2. A horizontal ventuirmeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of water. The pressure at inlet is 17.658 N/cm^2 and the vacuum pressure at the throat is 30 cm of mercury. Find the discharge of water through the ventuirmeter. Take C_d = 0.98.

3. Find the discharge of water flowing through a pipe of 30 cm diameter placed in an inclined position where a ventuirmeter is inserted, having a throat diameter of 15 cm. The difference of pressure between the main and throat is measured by a liquid of specific gravity 0.6 in an inverted U-tube which gives a reading of 30 cm. The loss of head between the main and throat is 0.2 times the kinetic head of the pipe.

4. In a 100 mm diameter horizontal pipe a ventuirmeter of 0.5 contrction ratio has been fixed. The head of water on the meter when there is no flow is 3 m (gauge). Find the rate of flow for which the throat pressure will be 2 m of water absolute. The coefficient of meter is 0.97. Take atmospheric pressure head = 10.3 m of water.

Linear Momentum equation

It is based on the law of conservation of momentum or momentum principle, which states that the net force acting on a fluid mass is equal to the change in momentum of the flow per unit time in that direction.

The force acting on a fluid mass ‘*m*’ is given by Newton’s second law of motion,

$$F = m\ x\ a$$

where ‘*a*’ is the acceleration acting in the same direction as force F.

But $$a = \frac{dv}{dt}$$

$$F = m\,\frac{dv}{dt}$$

(since mass is constant) $$F = \frac{d(mv)}{dt} \qquad (5)$$

Equation (5) is known as **Momentum principle** and it can also be written as

$$F.dt = d(mv) \qquad (6)$$

Equation (6) is known as Impulse-momentum equation and states that, the impulse of a force F acting on a fluid mass m in a short interval of time dt is equal to the change of momentum $d(mv)$ in the direction of force.

Applications of Momentum equation

It is used to determine:

1. The resultant force acting on the boundary of the flow passage, by a stream of fluid, as the stream changes its magnitude and direction or both. Problems of this type include:

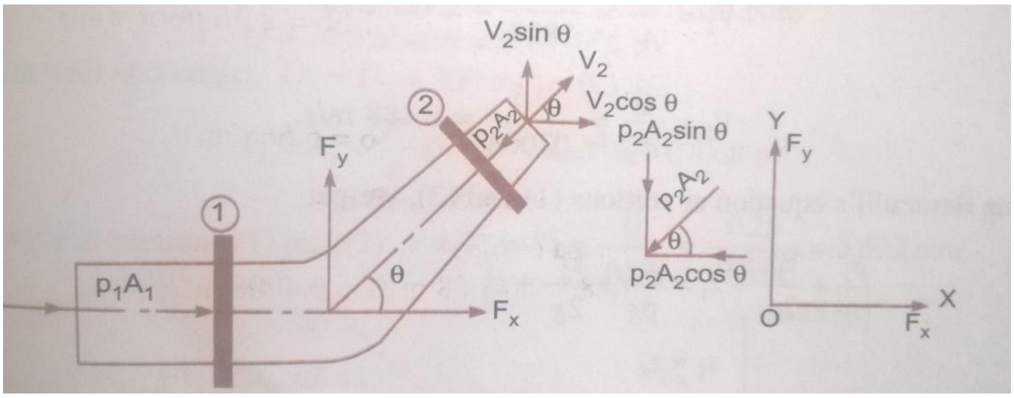
i. Pipe bends,
ii. Reducers
iii. Moving vanes
iv. Jet propulsions

2. The characteristic of the flow when there is an abrupt change of flow section.
For example: sudden enlargement in pipe, hydraulic jump in channel etc.

Forces on a Bend

The impulse momentum equation is used to determine the resultant force exerted by the flowing fluid on a pipe bend.

Consider two sections 1 and 2 as shown in the figure below.



Let V_1 = velocity of flow at section 1,

P_1 = pressure intensity at section 1,

A_1 = area of cross section of pipe at section 1 and

V_2, P_2, A_2 = corresponding values of velocity, pressure and area at section 2.

Let F_x and F_y be the components of the forces exerted by the flowing fluid on the bend in x and y directions respectively.

Then the force exerted by the bend on the flowing fluid will be equal to F_x and F_y but in the opposite directions. Hence component of force exerted by the bend on the flowing fluid in x-direction = - F_x and in the direction of y = - F_y.

The other forces acting on the fluid are P_1A_1 and P_2A_2 on the sections 1 and 2 respectively. Then momentum equation in x-direction is given by

Net force acting on the fluid in the direction of x = Rate of change of momentum in x-direction

$$P_1A_1 - P_2A_2 cos\theta - F_x = (mass\ per\ second)x(change\ of\ velocity)$$

$$= \rho\ Q(\text{final velocity in the direction of x – initial velocity})$$

$$= \rho Q(V_2 \cos\theta - V_1)$$

Therefore, $$F_x = \rho Q(V_1 - V_2 \cos\theta) + P_1A_1 - P_2A_2 cos\theta$$

Similarly, the momentum equation in y-direction gives

$$0 - P_2A_2 sin\theta - F_y = \rho Q(V_2 \sin\theta - 0)$$

Therefore, $$F_y = \rho Q(-V_2 \sin\theta) - P_2A_2 sin\theta$$

Now, the resultant force F_R acting on the bend is

$$F_R = \sqrt{F_x^2 + F_y^2}$$

And the angle made by the resultant force with horizontal direction is given by

$$\tan\theta = \frac{F_y}{F_x}$$

1. In a 45º bend a rectangular air duct of 1 m^2 cross-sectional area is gradually reduced to 0.5 m^2 area. Find the magnitude and direction of the force required to hold the duct in position if the velocity of flow at 1 m^2 section is 10 m/s and pressure is 2.943 N/cm^2. Take density of air as 1.16 kg/m^3.

Ans. Given data:

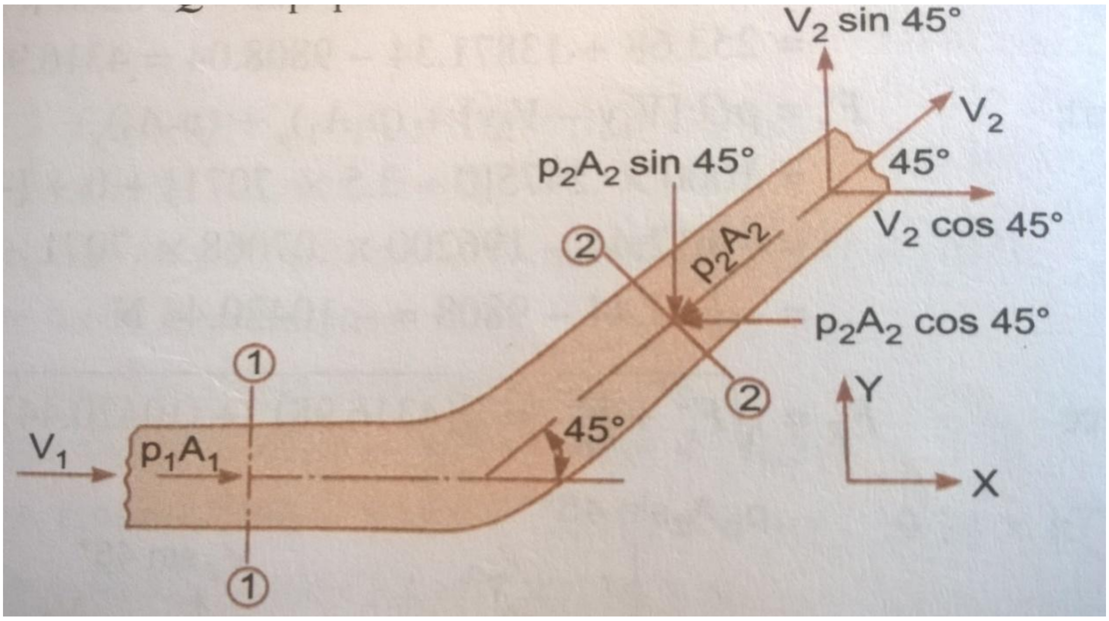
Area at section 1, A_1 = 1 m^2

Area at section 2, A_2 = 0.5 m^2

Velocity at section 1, $V_1 = 10$ m/s

Pressure at section 1, $P_1 = 2.943$ N/cm^2 = 2.943×10^4 N/m^2

Density of air = ρ = 1.16 kg/m^3



Applying continuity equation at sections 1 and 2,

$$A_1 V_1 = A_2 V_2$$

$$V_2 = \frac{1}{0.5} x \; 10 \; m/s = 20 \; m/s$$

and discharge $Q = A_1 V_1 = 10$ m^3/s

Applying Bernoulli's equation at sections 1 and 2,

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g}$$

$$\frac{2.943x\ 10000}{1.16\ x\ 9.81} + \frac{10^2}{2\ x\ 9.81} = \frac{p_2}{\rho g} + \frac{20^2}{2\ x\ 9.81} + 4$$

$$\frac{p_2}{\rho g} = \frac{2.943x\ 10000}{1.16\ x\ 9.81} + \frac{10^2}{2\ x\ 9.81} - \frac{20^2}{2\ x\ 9.81}$$

$$= 256.82 + 5.0968 - 20.387 = 2570.9 \text{ m}$$

Therefore, $p_2 = 2570.9\ x\ 1.16\ x\ 9.81 = 29255.8$ N

Force along x - axis, $F_x = \rho Q(V_{1x} - V_{2x}) + (P_1 A_1)_x + (P_2 A_2)_x$

where, $A_{1x} = 1$ m^2, $V_{2x} =$ V Cos 45° = 20 x 0.7071 m/s

$(P_1A_1)_x = 29430\ x\ 1 = 29430N$

$(P_2A_2)_x = -P_2A_2\text{Cos } 45º = -29255.8 \text{ x } 0.5\text{x } 0.7071$

Therefore, $F_x = 1.16\ x\ 10\ (10 - 20 \text{ x } 0.7071) + 29430 - 29255.8 \text{ x } 0.5\text{x } 0.7071$

= - 48.04 + 29430 - 10343.37 = - 19038.59 N

Force along y - axis, $F_y = \rho Q(V_{1y} - V_{2y}) + (P_1A_1)_y + (P_2A_2)_y$

where, $V_{1y} = 0$, $V_{2y} =$ V Sin 45° = 14.142 m/s

$(P_1A_1)_y = 0$

$(P_2A_2)_y = -P_2A_2\text{Sin } 45º = -29255.8 \text{ x } 0.5\text{x } 0.7071 = -10343.37\text{N}$

F_y = 1.16 x 10 (0 - 14.142) + 0 - 10343.37 = - 164.05 - 10343.37 = -10507.42 N

Therefore,

Resultant force, $F_R = \sqrt{F_x^2 + F_y^2}$

$= \sqrt{(19038.6)^2 + (10507.42)^2} = 21746.6\ N$

The direction of F_R with x - axis is given as,

$$\tan\theta = \frac{F_y}{F_x} = \frac{10507.42}{19038.6} = 0.5519$$

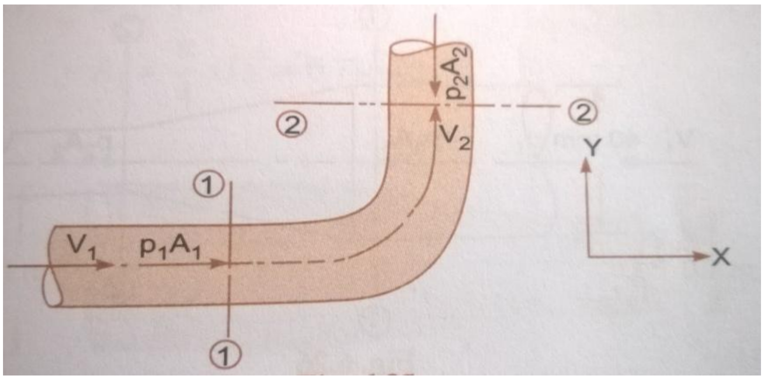
$$\theta = \tan^{-1}(0.5519) = 28°53'$$

2. A pipe of 300 mm diameter conveying 0.03 m^3/s of water has a right angled bend in a horizontal plane. Find the resultant force exerted on the bend if the pressure at the inlet and outlet of the bend are 24.525 N/cm^2 and 23.544 N/cm^2.

Ans. Given data:

Diameter of bend = D = 300 mm = 0.3 m

Area of the bend, A= $A_1 = A_2 = \frac{\pi}{4}0.3^2 = 0.07068\ m^2$



Velocity, $V_1 = V_2 = \frac{Q}{A} = \frac{0.3}{0.07068} = 4.244\ m/s$

Angle of bend = θ = 90º

Pressure at section 1, P_1 = 24.525 N/cm^2 = 24.525 x 10^4 N/m^2

P_2 = 23.544 N/cm^2 = 23.544 x 10^4 N/m^2

Force along x - axis, $F_x = \rho Q(V_{1x} - V_{2x}) + (P_1A_1)_x + (P_2A_2)_x$

where, ρ = 1000 kg/m^3, $V_1 = V_{1x}$ = 4.224 m/s, V_{2x} = 0

$(P_1A_1)_x = P_1A_1 = 245250\ x\ 0.07068\ N$

$(P_2A_2)_x = 0$

Therefore, $F_x = 1000\ x\ 0.30\ (4.244 - 0) + (245250\ x\ 0.07068) + 0$

= 1273.2 + 17334.3 = 18607.5 N

Force along y - axis, $F_y = \rho Q(V_{1y} - V_{2y}) + (P_1A_1)_y + (P_2A_2)_y$

where, V_{1y} = 0 , V_{2y} = V = 4.244 m/s

$(P_1A_1)_y = 0$

$(P_2A_2)_y = -P_2A_2 = -235440 \text{ x } 0.07068 = -16640.9$ N

F_y = 1000 x 0.30 (0 - 4.224) + 0 - 16640.9 = - 1273.2 - 16640.9 = - 17914.1 N

Therefore,

Resultant force, $F_R = \sqrt{F_x^2 + F_y^2}$

$= \sqrt{(18607.5)^2 + (17914.1)^2} = 21746.6\ N$

The direction of F_R with x - axis is given as,

$$\tan\theta = \frac{F_y}{F_x} = \frac{17914.1}{18607.5} = 0.9627$$

$$\theta = \tan^{-1}(0.9627) = 43°54'$$

Exercise problems

1. A 45º reducing bend is connected in a pipe line, the diameters of inlet and outlet of the bend being 600 mm and 300 mm respectively. Find the force exerted by the water on the bend if the intensity of pressure at inlet to bend is 8.829 N/cm^2 and rate of flow of water is 600 litres/sec.

2. 250 litres/sec of water is flowing in a pipe having a diameter of 300 mm. If the pipe is bent by 135º (that is change from initial to final direction is 135 º), find the magnitude and direction of the resultant force on the bend. The pressure of water flowing is 39.24 N/cm^2.

3. A 300 mm diameter pipe carries water under a head of 20 m with a velocity of 3.5 m/s. If the axis of the pipe turns through 45º, find the magnitude and direction of the resultant force at bend.

Moment of Momentum Equation

It is derived from the angular momentum principle which states that the resulting torque acting on a rotating fluid is equal to the rate of change of angular momentum or rate of change of moment of momentum.

Let V_1 = velocity of flow at section 1,

r_1 = radius of curvature at section 1 and

V_2, r_2 = corresponding values of velocity and radius of curvature at section 2.

Q = rate of flow of fluid and p = density of the fluid

Then moment of momentum per second of fluid at section 1 = $\rho Q V_1 r_1$

Similarly, moment of momentum per second of fluid at section 2 = $\rho Q V_2 r_2$

Therefore, rate of change of moment of momentum = $\rho Q V_2 r_2 - \rho Q V_1 r_1$

$$= \rho Q [V_2 r_2 - V_1 r_1]$$

According to moment of momentum principle, Resultant torque $\tau = \rho Q [V_2 r_2 - V_1 r_1]$.

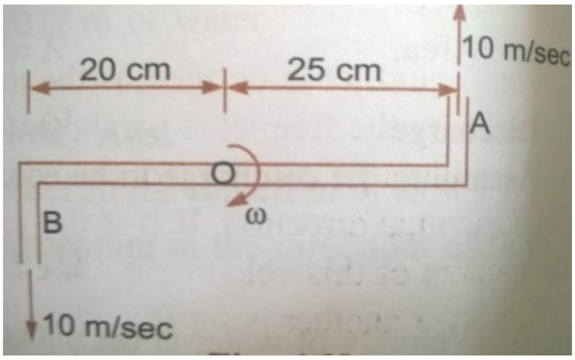
The above equation is known as **Moment of momentum equation**.

Applications of moment of momentum equation

This equation is applied:

1. For analysis of flow problems in turbines and centrifugal pumps
2. For finding torque exerted by water on the sprinkler.

1. A lawn sprinkler as shown in below figure has 0.8 cm diameter nozzle at the end of a rotating arm and discharges water at the rate of 10 m/s velocity. Determine the torque required to hold the rotating arm stationary. Also determine the constant speed of rotation of the arm, if free to rotate.



Ans.

Diameter of each nozzle = 0.8 cm = 0.008 m

Then, area of each nozzle, A = $\frac{\pi}{4} 0.008^2 = 0.00005026\ m^2$

Velocity of flow at each nozzle, V = 10 m/s

Then discharge through each nozzle, Q = AxV = 0.00005026 x 10 = 0.0005026 m^3/s

Torque exerted by water coming through nozzle A on the sprinkler = Moment of momentum of water through A,

$$= r_A\ x\ \rho\ x\ Q\ x\ V_A = 0.25\ x\ 1000\ x\ 0.0005026\ x\ 10\ clockwise$$

Torque exerted by water coming through nozzle B on the sprinkler,

$$= r_B\ x\ \rho\ x\ Q\ x\ V_B = 0.2\ x\ 1000\ x\ 0.0005026\ x\ 10\ clockwise$$

Therefore, Total Torque exerted by water on the sprinkler,

$$= (0.2\ x\ 1000\ x\ 0.0005026\ x\ 10) + (0.2\ x\ 1000\ x\ 0.0005026\ x\ 10)$$

= 1.2565 + 1.0052 = 2.26 N-m.

Speed of rotation of arm, if free to rotate

Let $\omega = speed\ of\ rotation\ of\ the\ sprinkler$

The absolute velocity of flow of water at the nozzles A and B are $V_1 = 10 - 0.25 \times \omega$

Torque exerted by water coming out at A, on sprinkler

$= r_A \times \rho \times Q \times V_1 = 0.25 \times 1000 \times 0.0005026 \times (10 - 0.2\ \omega)$

$= 0.12565\ (10 - 0.25\omega)$

Torque exerted by water coming out at B, on sprinkler

$$= r_B \times \rho \times Q \times V_2 = 0.20 \times 1000 \times 0.0005026 \times (10 - 0.2\omega)$$

$$= 0.10052\ (10 - 0.2\omega)$$

Therefore, total torque exerted by water = $0.12565\ (10 - 0.25\omega) + 0.10052\ (10 - 0.2\omega)$

Since moment of momentum of the flow entering is zero and no external torque is applied on sprinkler, so the resultant torque on the sprinkler must be zero.

Therefore, $0.12565\ (10 - 0.25\omega) + 0.10052\ (10 - 0.2\omega) = 0$

$$1.2565 - 0.0314\omega + 1.0052 - 0.0201\,\omega = 0$$

$$1.2565 + 1.0052 = \omega(0.0314 + 0.0201)$$

$$2.2617 = 0.0515\omega$$

Therefore, $\omega = \frac{2.2617}{0.0515} = 43.9\ {}^{rad}/_{sec}$

and $N = \frac{60\ x\ \omega}{2\pi} = \frac{60\ x\ 43.9}{2\pi} = 419.2\ r.p.m.$

Exercise problem

1 A lawn sprinkler of two nozzles of diameter 4 mm each is connected across a tap of water as show in below figure. The nozzles are at a distance of 30 cm and 20 cm from the centre of the tap. The rate of flow of water through the tap is 120 cm^3/s. The nozzles discharge water in the downward direction. Determine the angular speed at which the sprinkler will rotate free.